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On Conceptual Metaphors In Mathematics*

Abstract. This paper contains a few critical remarks concerning some fundamental assumptions and claims propagated by Lakoff and Núñez in their monograph Lakoff, Núñez (2000). Our attitude is skeptical (cf. also Pogonowski, 2011). We agree with the idea that *conceptual metaphors* may play some role in the formation of elementary mathematical notions. However, we disagree with the authors' claim that such metaphors provide the main mechanism in the emergence of new notions in advanced mathematics.

1. Main ideas of *Where Mathematics Comes From*

The monograph by Lakoff, Núñez (2000) tries to implement the solutions obtained in the famous text *Metaphors we live by* (cf. Lakoff, Johnson, 1980) into reflections regarding mathematics. The book was devoted to linguistics and without any doubt was a great success. One could justly suppose that, sooner or later, someone will try to use *conceptual metaphors* in the analysis of other symbolic systems. The first victim appears to be mathematics, but who knows – there is a number of symbolic systems not yet explored from this perspective. What about e.g. *embodied theoretical physics*?

1.1. Embodied cognition

Embodied philosophy is a clearly distinguished trend in epistemology (cf. for instance Lakoff, Johnson, 1999). It is related to the old mind-body problem as well as to enactivism. The program, entitled *embodied mathematics*, is declared to be a cognitive approach to mathematics. It is thus neither a part of mathematics itself nor a standpoint in the philosophy of mathematics. Being an external view

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on mathematics, it should, nevertheless, keep track of the mathematical practice of professional mathematicians; otherwise it cannot claim to have any explanatory power as far as the actually functioning mathematics is concerned. Furthermore, claiming neutrality with respect to the main standpoints in the philosophy of mathematics (as the authors do) is not compatible with their emphatic critique of Platonism.

Embodied mathematics situates itself in the second generation of the investigations of artificial intelligence and rejects the previous approach, i.e. computationalism. The essence of embodied mathematics appears to be the (main or even sole) role ascribed to conceptual metaphors in the formation and functioning of mathematics.

Embodied philosophy stresses the fact that the role of the brain should not be compared to that of a computing device. It should rather be conceived as a device which enables the survival of the organism via its reactions to the environment. Embodied philosophy tries to incorporate the results of empirical experiments concerning human cognition into its general claims.

1.2. Conceptual metaphors: a general scheme

Conceptual metaphors may be shortly characterized as mappings which preserve information (cf. the definition below). Concepts from the source domain are used in the formation of new concepts in the target domain, and the relations between them in the target domain are projections of the corresponding relations in the source domain.

One may justifiably ask what the difference is between conceptual metaphors and the results of analogy-based reasoning. The authors claim that conceptual metaphors create new concepts, while analogy is responsible for the comparison of already existing concepts. This is a subtle matter. If conceptual metaphor is a kind of homomorphism, then its domain and counterdomain should be provided in advance.

In order to exclude misunderstandings we quote the original definition of a conceptual metaphor proposed by the authors (Lakoff, Núñez, 2000, p. 6):

> Conceptual metaphor is a cognitive mechanism for allowing us to reason about one kind of thing as if it were another. [...] It is a grounded, inference-preserving cross-domain mapping – a neural mechanism that allows us to use the inferential structure of one conceptual domain (say, geometry) to reason about another (say, arithmetic).

The authors explicitly mention the following three main insights concerning the nature of the mind:

1. Human reason and concepts are structured by the nature of our bodies, brains, and everyday functioning.
2. Most thought is unconscious, inaccessible to direct introspection. We do not have access to our low-level thought processes.

3. Humans conceptualize abstract concepts in concrete terms. Essential in this respect are ideas and modes of reasoning grounded in the sensory-motor system.

They also stress that all these insights are relevant to mathematical thinking. Consequently, the formation of mathematical concepts could be ultimately reduced to ideas which are grounded in the sensory-motor system, while mathematical argumentation could be ultimately based on unconscious thought processes. Such extreme claims do not seem to adequately describe the mathematical abilities and activities of humans.

1.3. Conceptual metaphors in mathematics

The authors claim the existence of two main types of conceptual metaphors in mathematics:

1. *Grounding metaphors.* They yield basic, directly grounded ideas (e.g.: addition as adding objects to a collection, sets as containers).
2. *Linking metaphors.* They yield sophisticated (abstract) ideas. Examples are: numbers as points on a line, geometrical figures as algebraic equations.

Cognitive scientists make essential use of several *image schemas*, understood as recurring structures in human cognitive processes. They are responsible for establishing patterns of understanding the concepts and reasoning behind them. Their origin is linked to language use, physical experience and cultural heritage. Examples of image schemas are: containment, source-path-goal, center-periphery, rotation, contact, equilibrium, etc.

A *conceptual blend* is understood as a combination of two distinct cognitive structures with fixed correspondences between them. If this correspondence is based on a conceptual metaphor, it is a *metaphorical blend.* As prominent example of such a blend is the Number-Line Blend, which is based on a correspondence provided by the Numbers Are Points On Line metaphor.

The authors list the following four grounding conceptual metaphors which, according to them, are crucial in arithmetic:

1. *Arithmetic as object collection.*
2. *Arithmetic as object construction.*
3. *The measuring stick metaphor.*
4. *Arithmetic as motion along a path.*

These metaphors are grounding, because each of them provides a direct link from the domain of a sensory-motor experience to the corresponding mathematical domain, in this case the domain of natural numbers. The names of these metaphors are self-explanatory.

The most important conceptual metaphor used extensively by the authors is the *Basic Metaphor of Infinity* (shortly: BMI). They introduce it in the following way (Lakoff, Núñez, 2000, p. 158):

> We hypothesize that all cases of actual infinity – infinite sets, points at infinity, limits of infinite series, infinite intersections, least upper bounds – are special cases of a single general conceptual metaphor in which processes that go on indefinitely are conceptualized as having an end and an ultimate result. We call this metaphor the *Basic Metaphor of Infinity*, or the BMI for short. The target domain of the BMI is the domain of processes without end – that is, what linguists call imperfective processes. The effect of the BMI is to add a metaphorical completion to the ongoing process so that it is seen as having a result – an infinite *thing*.

Te applications of the Basic Metaphor of Infinity are ubiquitous in the Lakoff, Núñez (2000) monograph. According to the authors, mathematicians employ this metaphor in all limitative processes, in the introduction of several types of infinitary objects, in reasoning by induction, in applying closure rules, etc.

The authors try to illustrate the method of forming conceptual metaphors by providing numerous examples related to elementary arithmetic, algebra, geometry, and analysis. All these examples repeat the same pattern: some new concepts are formed as the results of a transition from a source domain into the corresponding target domain. The reader might be under the impression that mathematics as a whole is a hierarchy of concepts developed in this way.

1.4. Philosophical conclusions

Despite the declaration of neutrality with respect to the existing standpoints in the philosophy of mathematics, the authors formulate several declarations of a clearly philosophical nature. The two most important such declarations are most likely the following:

1. Embodied mathematics is sufficient for the abolition of the *romance of mathematics* (authors' term), i.e. the view that mathematics has an objective existence and that human mathematics allows us to discover truths about the world.
2. Human mathematics is the whole of mathematics, i.e. there does not exist any transcendental mathematics, independent of human minds.

The authors claim that the problem of the nature of human mathematics is empirical, and thus neither mathematical nor philosophical. It should be investigated by cognitive science, which looks for relations between the brain and the mind.

2. General critical remarks

Our criticism in this section will be limited to the mathematical inaccuracies and philosophical implications of the approach in question.

2.1. Mathematical inaccuracies

We focus our attention on a section of the discussed book devoted to sets and infinities. More critical remarks (e.g. concerning the authors' approach to the work of Dedekind and Weierstrass) can be found in Pogonowski (2011).

2.1.1. Set theory

A typical representation of a conceptual metaphor in the book takes the form of a table consisting of columns, corresponding, to the source domain and to the target domain, respectively. For example, the Classes Are Containers metaphor is represented as follows (Lakoff, Núñez, 2000, p. 123):

Source domain Container schemas		*Target domain* Classes
Interiors of Container schemas	→	Classes
Objects in interiors	→	Class members
Being an object in an interior	→	The membership relation
An interior of one Container schema within a larger one	→	A subclass in a larger class
The overlap of the interiors of two Container schemas	→	The intersection of two classes
The totality of the interiors of two Container schemas	→	The union of two classes
The exterior of a Container schema	→	The complement of a class

The authors claim that this is our natural, everyday unconscious conceptual metaphor for what a class is and that it grounds our concept of class in our concept of a bound region in space. These claims may perfectly fit into our intuitions concerning small finite sets (classes) of objects. However, they seem to be misleading in the case of arbitrary infinite sets. Let us mention that the first applications of the general concept of a set, due to Cantor, were devoted first of all to rather complicated sets of real numbers. Another problem concerning our intuitive understanding of collections of objects is the fundamental distinction between the collective and distributive meaning of the concept of a set. Contemporary mathematics accepted the latter, but teachers of mathematics report the difficulties of students in regard to grasping a clear distinction between these two meanings (cf. e.g. Bryll, Sochacki, 2009, p. 267–275).

The authors present a few elementary set theoretical concepts in a rather loose way. They do not include the axiom schema of replacement on their list of axioms of set theory. This is surprising, as without this axiom one is unable to perform constructions based on transfinite induction, which are fundamental in set theory. Let us recall that the axiom schema of replacement states – in an intuitive formulation – that for any function, the image of any subset of its domain is a set. Without this axiom, one cannot prove, for instance, that $\aleph_\omega$ is a set, which was already noted by Abraham Fraenkel.

The reader might be under the impression that infinite cardinal numbers emerge only as the result of the iterations of the operation of taking the power set of a countable set, which is evidently not the case. This way (also making use of the operation of union at limit steps) one obtains the beth cardinal numbers $\beth_\alpha$.

The transfinite hierarchy of alephs $\aleph_\alpha$ is defined in a different way. There are, in fact, several ways of introducing cardinal numbers. An old idea (known to Cantor and Frege) to identify the cardinal number of a set X with the class of all sets equinumerous with X has to overcome the difficulty that such a class is not a set. In the presence of the axiom of choice, one can define the cardinal number of a set X as the least ordinal number α such that there exists a bijection between X and α. Let us also recall that in 1915 Hartogs proved that there is a least well-ordered cardinal greater than a given well-ordered cardinal (Hartogs, 1915). By the Hartogs number of a set X we mean the least ordinal α such that there is no injection from α into X. This idea can be then used to define the hierarchy of alephs. The generalized continuum hypothesis which claims that the hierarchies of beth numbers and alephs coincide is independent of the axioms of the Zermelo-Frankel set theory.

There are several reasons as to why modern set theory investigates large cardinal axioms (e.g. strongly inaccessible cardinals, measurable cardinals, etc.), and partly purely mathematical (e.g. partition theorems), partly metatheoretical (concerning proofs of relative consistency). We consider it doubtful that all of these reasons could be based on conceptual metaphors alone.

In our opinion it is not the Classes Are Containers metaphor which is a fundamental intuition concerning sets. Rather, the very idea of well-foundation and a possibility of forming a transfinite hierarchy of sets play a crucial role in set theory.

2.1.2. Infinity

Let us consider a typical example of how the authors make use of their Basic Metaphor of Infinity (Lakoff, Núñez, 2000, p. 174, *The Set of All Natural Numbers*):

Target Domain Iterative Processes That Go On and On		*Special Case* The Set of Natural Numbers
The beginning state (0)	$\Rightarrow$	The natural number frame, with a set of existing numbers and a successor operation that adds 1 to the last number and forms a new set
State (1) resulting from the initial stage of the process	$\Rightarrow$	The empty set, the set of natural numbers smaller than 1
The process: From a prior intermediate state ($n-1$), produce the next state (n).	$\Rightarrow$	Given S_{n-1}, the set of natural numbers smaller than $n-1$, form $S_{n-1} \cup \{n-1\} = S_n$.
The intermediate result after that iteration of the process (the relation between n and $n-1$)	$\Rightarrow$	At state n, we have S_n, the set of natural numbers smaller than n.
"The final resultant state" (actual infinity "∞")	$\Rightarrow$	**S_∞, the set of *all* natural numbers smaller than ∞ – that is, the set of *all* natural numbers (which does not include ∞ as a number)**
Entailment *E*: The final resultant state ("∞") is unique and follows every nonfinal state.	$\Rightarrow$	**Entailment *E*: The set of all natural numbers is unique and includes every natural number (no more, no less).**

The authors claim that this metaphor does the same work in cognitive perspective as the axiom of infinity does in set theory. In our opinion, this declaration is misleading. The axiom of infinity in set theory implies the existence of at least one infinite set. This set has $\emptyset$ as its element, and if x belongs to this set, then $x \cup \{x\}$ also belongs to it (sets with these properties are called inductive sets). The smallest set of this kind is the set of all finite von Neumann ordinal numbers and may represent the universe of the standard model of Peano arithmetic. One can prove that such a set exists: if there exists at least one inductive set, then there exists the smallest inductive set.

The standard model of arithmetic (having exactly the natural numbers as its universe, "no more, no less") can be uniquely characterized (i.e. up to isomorphism) in the second-order language but it cannot be uniquely characterized in the first-order language. We mention this fact, because many applications of the BMI presented by the authors implicitly assume that the limit object allegedly existing as the result of this process is unique. Needless to say, mathematicians do not work in this way: the introduction of an object *requires the proof of its existence as well as its uniqueness.*

Another surprising application of the BMI can be found in an article by Núñez, where the author tries to convince us that a circle is an infinitary object conceived as a limit of a sequence of regular polygons (Núñez, 2005, p. 1772):

> A case of actual infinity: the sequence of regular polygons with n sides, starting with $n = 3$ (assuming that the distance from the center to any of the vertices is constant). The sequence is endless but it is conceived as being completed. The final resultant state is a very peculiar entity, namely, a circle conceived as a polygon with infinitely many sides of infinitely small magnitude.

The conclusion of this quotation resembles a fairy tale trick. The mathematical concept of a circle is surely not introduced in the aforementioned way. A circle is defined in terms of (a suitably chosen) distance function. Circles "look differently" in the Euclidean metric, Manhattan metric, Tchebyshev metric, etc. Moreover, one may notice here that the author considers the non-denumerable set of all points forming the circle as the limit of a sequence of finite collections of points (the vertices of the polygons in question).

The Basic Metaphor of Infinity neither implies the existence of the desired infinitary object nor justifies its uniqueness. For example, let us imagine that we want to form a *The Most Slowly Divergent Series* concept. It is possible to provide a precise definition of what is meant by saying that one series diverges more slowly than the other. However, there does not exist a most slowly divergent series. Let $(a_n)_{n\in\mathbb{N}}$ be a decreasing sequence of positive real numbers. If the series $\sum_{n=1}^{\infty} a_n$ is divergent, then the series $\sum_{n=1}^{\infty} \frac{a_n}{s_n}$ is divergent as well, where $s_n = \sum_{k=1}^{n} a_k$. Of course, one may metaphorically consider the (sic!) most slowly divergent series, but one should be aware that such a discussion does not belong to mathematics.

2.1.3. Other inaccuracies

We are not going to discuss all of the inaccuracies observed in Lakoff, Núñez (2000). Let us only mention a few of them:

1. The concept of *granular numbers* is vague. The authors try to explicate the mathematical representation of infinitesimals proposed in non-standard analysis, but their attempt is flawed. The reader might be under the impression that infinitesimals exist in the standard field $\mathbb{R}$ of real numbers. The construction of the hyperreal field (based on the notion of ultraproduct) is omitted. Indeed, one may justly ask as to what kind of a conceptual metaphor could be responsible for creating the concept of *equality almost everywhere*.
2. When discussing points in infinity in projective geometry the authors try to introduce them by another application of their favorite Basic Metaphor of Infinity, illustrating this process by a series of isosceles triangles with longer and longer sides. However, their approach does not reflect the method of introducing ideal elements as proposed by Hilbert (cf. Hilbert, 1926).
3. Remarkably, the authors do not apply their method of conceptual metaphors to concepts from general topology. We dare to claim that several topological concepts escape the possibility of being captured by a conceptual metaphor. This concerns firstly, pathological objects constructed on purpose, with the aim of sharpening existing topological intuitions. Looking for counterexamples is a standard mathematical procedure, as it enables one to explicitly reveal the role played by the assumptions of a theorem, as well as shows the restrictions imposed on the use of mathematical concepts, etc. This concerns all branches of mathematics – cf. e.g. Gelbaum, Olmsted, 2003; Steen, Seebach, 1995; Wise, Hall, 1993.

2.2. Philosophical doubts

The authors' philosophical declarations also raise some doubts. Let us mention a few of them:

1. We doubt that the rejection of Platonism in mathematics on the grounds of the embodied mathematics program is justified. We dare to suggest a modest approach, a kind of *mathematical agnosticism*: there may exist transcendental mathematics, but its existence (or non-existence) does not influence the work of professional mathematicians.
2. The authors' claim that human mathematics exhausts the whole of mathematics is, in our opinion, not justified as well. As an argument against this claim, one can take the *unreasonable effectiveness of mathematics in the natural sciences*. In particular, the fact that pure mathematical results sometimes precede their counterparts in physics suggests a kind of dependence of human mathematics on the external mathematical aspects of Nature.

3. Mathematics, as described by the authors, is restricted to mathematics presented in elementary textbooks. It seems that they checked the definitions of chosen elementary mathematical notions provided in the textbooks and then tried to impose a net of image schemas, conceptual metaphors, metaphorical blends, etc. on them. The entire context of mathematical discovery is lost.
4. It is unclear as to how the embodied mathematics program could explain the changes in mathematical intuitions. Unlike intuitions based on every-day experience, mathematical intuitions are dynamic. There are several reasons for their change: the development of mathematical knowledge, the resolution of paradoxes, purposively-formulated research programs, choices of aesthetic values, the mathematical fashion of a given epoch, etc. We seriously doubt that all of these factors can be embraced by an approach based on conceptual metaphors, which ultimately relates everything to the grounding sensory-motor factors.
5. It is not clear to us as to how the formation process of conceptual metaphors could explain the *emergence* of sophisticated mathematical notions (e.g.: randomness, equality almost everywhere, compactness, exotic sphere, etc.). Obviously, if a concept has already been formed, then one may try to build a more-or-less sophisticated conceptual metaphor leading to it, either directly or indirectly, in multiple stages.
6. Embodied mathematics is helpless with respect to the incompleteness phenomena in mathematics – it does not provide any hint as to which choices should be made with respect to statements independent of the accepted axioms. Similarly, conflicts of intuitions independently supported by incompatible mathematical arguments (e.g. the axiom of choice and the axiom of determinacy) cannot be resolved using conceptual metaphors.
7. Concept formation is only one of many activities performed by professional mathematicians in the process of creating mathematics. There are many other such activities, e.g.: abstraction, generalization, reasoning by analogy, induction or abduction, looking for counterexamples, etc. Most importantly, the main mathematical activity concerns *proving* theorems. We see no possibility of representing the latter activity in terms of conceptual metaphors. *Metaphorical deduction* does not exist, as it is an inconsistent notion.
8. The formation of conceptual metaphors is based on language. The difference between defining and describing mathematical objects is not reflected in this procedure. Furthermore, one can show that some concepts cannot be defined in a given language (e.g. the concept of a sentence true in the standard model of arithmetic is not definable in the language of arithmetic itself). How would the authors respond to this fact?

3. Reviews of Lakoff, Núñez 2000

The monograph Lakoff, Núñez (2000) has been reviewed by several authors; cf. e.g.: Auslander, 2001; Brożek, Hohol, 2014 (chapter II); Devlin, 2008; Elglaly, Quek, 2009; Gold, 2001; Goldin 2001; Henderson, 2002; Madden, 2001; Siegfried, 2001; Schiralli, Sinclair, 2003; Voorhees, 2004. The reviews are mostly critical, pointing to mathematical inaccuracies and limited explanatory power as far as real mathematical practice of professional mathematicians is concerned. Let us include here a short passage from Devlin 2008 in which the author presents some doubts about the role of cognitive metaphors in the learning of mathematics:

> Rather, a mathematician (at least me and the others I've asked) learns new math the way people learn to play chess. We first learn rules of chess. Those rules don't relate to anything in our everyday experience. They don't make sense. They are just rules of chess. To play chess, you don't have to understand the rules or know where they came from or what they "mean". You simply have to follow them. In our first few attempts at playing chess, we follow the rules blindly, without any insight or understanding what we are doing. And, unless we are playing another beginner, we get beat. But then, after we've played a few games, the rules begin to make sense to us – we start to *understand* them. Not in terms of anything in the real world or in our prior experience, but in terms of the game itself. Eventually, after we have played many games, the rules are forgotten. We just play chess. And it really does make sense to us. The moves do have meaning (in terms of the game). But this is not a process of constructing a metaphor. Rather it is one of *cognitive bootstrapping* (my term), where we make use of the fact that, through conscious effort, the brain can learn to follow arbitrary and meaningless rules, and then, after our brain has sufficient experience working with those rules, it starts to make sense of them and they acquire meaning for us. (At least it does if those rules are formulated and put together in a way that has structure that enables this.)

We also share the opinion expressed by Madden (Madden, 2001, p. 1187):

> If I think about the portrayal of mathematics in the book as a whole, I find myself disappointed by the pale picture the authors have drawn. In the book, people formulate ideas and reason mathematically, realize things, extend ideas, infer, understand, symbolize, calculate, and, most frequently of all, *conceptualize.* These plain vanilla words scarcely exhaust the kinds of things that go on when people do mathematics. They explore, search for patterns, organize data, keep track of information, make and refine conjectures, monitor their own thinking, develop and execute strategies (or modify or abandon them), check their reasoning, write and rewrite proofs, look for and recognize errors, seek alternate descriptions, look for analogies, consult one another, share ideas, encourage one another, change points of view, learn new theories, translate problems from one language to another, become obsessed, bang their heads against walls, despair, and find light. Any one of these activities is itself enormously complex cognitively – and in social, cultural, and historical dimensions as well. In all this, what role metaphors play?

> Moving to a different perspective, I want to note that there are areas not even hinted at in the book where cognitive science is prepared to contribute to our understanding of mathematical thought. Consider this: Metaphorical ideas are frequently misleading, sometimes just plain wrong. Zariski spent most of his career creating a precise language and theory capable of holding the truths that the Italian geometers had glimpsed intuitively while avoiding the errors into which they fell. What cognitive mechanism enable people to recognize that a metaphor is not doing the job it is supposed to do and to respond by fashioning better conceptual tools? [...]
>
> If mathematical thinking is like other kinds of thinking in its use of metaphors, what distinguishes mathematical thinking may be the exquisite, conscious control that mathematicians exercise over how intuitive structures are used and interpreted. We can step back from our own thinking and critically examine our attempts at meaning-making. This, I would venture, is as fundamental a cognitive mechanism as any mentioned by Lakoff and Núñez.

4. Mathematical education

Is the embodied mathematics program dangerous for mathematical education? Well, there is nothing principally wrong in supporting the explanations of new mathematical concepts with intuitive comments, diagrams, references to motion, etc. However, the teacher should carefully distinguish between the formal definition and such intuitive comments. Let us look at the authors' introduction of The Ordered Pair Metaphor (Lakoff, Núñez, 2000, p. 141):

> Intuitively, an ordered pair is conceptualized nonmetaphorically as a subitized pair of elements (by what we will call a Pair schema) structured by a Path schema, where the source of the path is seen as the first member of the pair and the goal of the path is seen as the second member. This is simply our intuitive notion of what an ordered pair is.
>
> With the addition of the Sets Are Objects metaphor, we can conceptualize ordered pairs metaphorically, not in terms of Path and Pair schemas but in terms of sets:

The Ordered Pair Metaphor

Source Domain Sets		*Target Domain* Ordered Pairs
The Set $\{\{a\}, \{a, b\}\}$	$\rightarrow$	The Ordered Pair (a, b).

> Using this metaphorical concept of an ordered pair, one can go on to metaphorically define relations, functions, and so on in terms of sets.

We think that the teaching practice in this respect is completely different. Obviously, we start with some intuitive comments, pointing to the fact that the sets $\{a, b\}$ and $\{b, a\}$ are identical (because of the axiom of extensionality) followed by the formulation of our task: how to define a construct in which the ordering of its elements is fixed. Then, we propose the definition: $(a, b) =_{df} \{\{a\}, \{a, b\}\}$. Then, we prove that $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$, and we are finished. There is no need to discuss paths, sources, schemas, blends, etc.

What could be said about the possible impact of the embodied mathematics project on the working practice of professional mathematicians? We think that they will treat the project as a curiosity rather than a collection of hints on how to develop mathematics. As an example, let us compare e.g. the views of Thurston on understanding the concept of a derivative (Thurston, 1994); please notice that he disagrees with the claims of the proponents of the embodied mathematics program:

> People have different ways of understanding particular pieces of mathematics. To illustrate this, it is best to take an example that practicing mathematicians understand in multiple ways, but that we see our students struggling with. The derivative of a function fits well. The derivative can be thought of as:
>
> 1. Infinitesimal: the ratio of the infinitesimal change in the value of a function to the infinitesimal change in a function.
> 2. Symbolic: the derivative of x^n is nx^{n-1}, the derivative of $\sin(x)$ is $\cos(x)$, the derivative of $f \circ g$ is $f' \circ g * g'$, etc.
> 3. Logical: $f'(x) = d$ if and only if for every ε there is a δ such that when $0 < |\triangle x| < \delta$,
>
> $$\left| \frac{f(x + \triangle x) - f(x)}{\triangle x} - d \right| < \delta.$$
>
> 4. Geometric: the derivative is the slope of a line tangent to the graph of the function, if the graph has a tangent.
> 5. Rate: the instantaneous speed of $f(t)$, when t is time.
> 6. Approximation: The derivative of a function is the best linear approximation to the function near a point.
> 7. Microscopic: The derivative of a function is the limit of what you get by looking at it under a microscope of higher and higher power.
>
> This is a list of different ways of *thinking about* or *conceiving of* the derivative, rather than a list of different *logical definitions*. Unless great efforts are made to maintain the tone and flavor of the original human insights, the differences start to evaporate as soon as the mental concepts are translated into precise, formal and explicit definitions.
>
> [...]
>
> These differences are not just a curiosity. Human thinking and understanding do not work on a single track, like a computer with a single central processing unit. Our brains and minds seem to be organized

> into a variety of separate, powerful facilities. These facilities work together loosely, "talking" to each other at high levels rather than at low levels of organization.

Notice that the last sentence is in contradiction with the claim that mathematical reasoning is mostly unconscious, and not accessible to direct introspection. It should be stressed that this is the opinion of a famous mathematician presumably based on his own research practice.

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